Linear Algebra I
Exercise Sheet II: Matrix Representations

Euclidean Space

Question 1. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$L(x, y) = (2x - y, 3y, x + 4y).$$

- Find the matrix representation of L with respect to the standard basis.
- Find the image of $(2, 1)$ under this map.
- Determine the columnspace of L.
- Find the image of the standard basis vectors of $\mathbb{R}^2$ under L. Find the image under L^2. Find the image under L^3.

Question 2. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$L(x, y) = (-x + 3y, 2x + y, 5x).$$

- Find the matrix representation of L with respect to the standard basis.
- Find the image of $(1, -1)$ under this map.
- Determine the columnspace of L.
- Find the image of the standard basis vectors of $\mathbb{R}^2$ under L. Find the image under L^2. Find the image under L^3.

Question 3. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$L(x, y, z) = (x + y, 2y - z).$$

- Find the matrix representation of L with respect to the standard basis.
- Find the image of $(-1, 2, 0)$ under this map.
- Determine the columnspace of L.
- Find the image of the standard basis vectors of $\mathbb{R}^2$ under L. Find the image under L^2. Find the image under L^3.

Question 4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L(x, y, z) = (3x - y, x + 2z, 4y - z).$$

- Find the matrix representation of L with respect to the standard basis.
- Find the image of $(2, 3, -2)$ under this map.
- Determine the columnspace of L.
- Find the image of the standard basis vectors of $\mathbb{R}^2$ under L. Find the image under L^2. Find the image under L^3.

Question 5. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$L(x, y) = (x, x - y, 2x + 3y).$$

- Find the matrix representation of L with respect to the standard basis.
- Find the image of $(1, 17)$ under this map.
- Determine the columnspace of L.
- Find the image of the standard basis vectors of $\mathbb{R}^2$ under L. Find the image under L^2. Find the image under L^3.

Non-Euclidean Space

Question 6. Let V_n be the space of all polynomials of degree less than n over the rationals,

$$V_n = \{a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} \,|\, a_i \in \mathbb{Q}\}.$$

- Check that $\{1, x, x^2, x^3\}$ is a basis for V_4.
- Check whether $\{1, x-1, (x+1)^2, (x-1)(x+1)^2\}$ is a basis for V_4.
- Why is $(x-1)(x+1)^2$ not a linear combination of $(x-1)$ and $(x+1)^2$?

Let $L : V_4 \to V_4$ be given by

$$L\big(P(x)\big) = \frac{\mathrm{d}^2 P(x)}{\mathrm{dx}^2} + P(x).$$

Find the matrix representation of L with respect to the basis $\{1, x, x^2, x^3\}$.

Question 7. Let $\mathbb{M}_n$ be the vector space of $n \times n$ matrices with entries in $\mathbb{R}$. Let $L : \mathbb{M}_2 \to \mathbb{M}_2$ be given by

$$L(M) = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} M - M.$$

- Show that
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
is a basis of $\mathbb{M}_2$.
- Find the matrix representation of L with respect to this basis.

Question 8. This question is best attempted with the help of a computer algebra package like maple or mathematica. Also possible to use is python or C. One can attempt this question by hand, though I would advise against it. Let $\mathbb{M}_n$ be the vector space of $n \times n$ matrices with entries in $\mathbb{C}$. Let $L : \mathbb{M}_3 \to \mathbb{M}_3$ be given by

$$L(M) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} M - M.$$

- Find a basis for $\mathbb{M}_3$.
- Find the matrix representation of L with respect to this basis.